Math 228: Solutions for Problem Set Nine

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1. **Page 113, number 12**

   (d) We'll begin by using the Euclidean Algorithm to find the gcd.

$$\begin{aligned} 1575 &= 6(231) + 189 \\ 231 &= 1(189) + 42 \\ 189 &= 4(42) + 21 \\ 42 &= 2(21) + 0 \end{aligned}$$

   So the greatest common denominator of 1575 and 231 is 21. Now we'll go back up the list of equations to produce a linear combination of 1575 and 231 equaling 21. In other words, we are looking for integers m and n such that $m(1575) + n(231) = 21$.

$$\begin{aligned} 21 &= 189 - 4(42) \\ &= 189 - 4(231 - 1(189)) \\ &= 5(189) - 4(231) \\ &= 5(1575 - 6(231)) - 4(231) \\ &= 5(1575) - 34(231) \end{aligned}$$

   (h) Using the same technique as part d:

$$\begin{aligned} 100996 &= 5(20048) + 756 \\ 20048 &= 26(756) + 392 \\ 756 &= 1(392) + 364 \\ 392 &= 1(364) + 28 \\ 364 &= 13(28) + 0 \end{aligned}$$

So $gcd(100996, 20048) = 28$.

$$\begin{aligned}
28 &= 392 - 1(364) \\
&= 392 - 1(756 - 1(392)) \\
&= 2(392) - 1(756) \\
&= 2(20048 - 26(756)) - 1(756) \\
&= 2(20048) - 53(756) \\
&= 2(20048) - 53(100996 - 5(20048)) \\
&= 267(20048) - 53(100996)
\end{aligned}$$

2. **Page 112, number 7**

   There are two approaches to solving this problem. This problem can be solved using simultaneous equations to eliminate the x's. For example, since $a \mid 11x + 3$, there exists an integer m such that $ma = 11x + 3$ (this is one of the equations). However, it is easier to solve with the Euclidean Algorithm.

   By definition of gcd, if $a \mid b$ and $a \mid c$ then $a \mid gcd(b, c)$. Since $a \mid 11x + 3$ and $a \mid 55x + 52$, then a divides $gcd(55x + 52, 11x + 3)$.

   $$55x + 52 = 5(11x + 3) + 37$$

   From Euclid's Algorithm, $gcd(55x + 52, 11x + 3) = gcd(11x + 3, 37)$. So then $a \mid gcd(11x+3, 37)$. That means $a \mid 37$. But 37 is prime, so its only divisors are 1 and 37. Since $a > 1$, $a = 37$.

3. **Page 113, number 10**

   (b) Counter example: $4 \mid 12$ and $6 \mid 12$ but $4 \cdot 6 = 24 \nmid 12$.

   (c) Since $a \mid b$, there exists $k \in \mathbb{N}$ s.t. $k \cdot a = b$.
   Likewise, $a \mid c$ implies $\exists\ l \in \mathbb{N}$ s.t. $l \cdot a = c$.

   If we multiply the two equalities we get $(k \cdot a)(l \cdot a) = (b)(c)$.
   Rearranging terms: $(k \cdot l \cdot a)(a) = b \cdot c$. But since k, l and a are integers, their product is an integer. So $a \mid (b \cdot c)$.

4. **Page 113 number 10**

   (d) Since $a \mid b$, there exists $k \in \mathbb{N}$ s.t. $k \cdot a = b$.
   Likewise, $c \mid d$ implies $\exists\ l \in \mathbb{N}$ s.t. $l \cdot c = d$.

   If we multiply the two equalities we get $(k \cdot a)(l \cdot c) = (b)(d)$. Rearranging terms: $(k \cdot l)(a \cdot c) = b \cdot d$. But since k and l are integers, their product is an integer. So $(a \cdot c) \mid (b \cdot d)$.

   (e) Since $a \mid b$, $\exists k \in \mathbb{N}$ s.t. $k \cdot a = b$. So $\frac{b}{a}$ is an integer and $\frac{b}{a} = k$. Since $c \mid \frac{b}{a}$, $\exists\ l \in \mathbb{N}$ such that $l \cdot c = \frac{b}{a}$. Multiplying by a:

   $$l \cdot c \cdot a = b$$

   i. Because l and a are both integers, their product is an integer. So $(l \cdot a) \cdot c = b$ implies that c divides b. Now we know that $\frac{b}{c}$ is an integer.

   ii. We can rearrange $l \cdot c \cdot a = b$ by dividing by c (we know $c \neq 0$ because $c \mid k$).

   $$l \cdot a = \frac{b}{c}$$

   Since l is an integer, $a \mid \frac{b}{c}$.

5. **Page 113 number 20**

   We will use the Euclidean Algorithm to find the gcd:

   $$\begin{aligned} 5k+3 &= 1(3k+2)+(2k+1) \\ 3k+2 &= 1(2k+1)+(k+1) \\ 2k+1 &= 1(k+1)+k \\ k+1 &= 1(k)+1 \\ k &= k(1)+0 \end{aligned}$$

   So the gcd of 5k+3 and 3k+2 is 1.