# Math 228: Solutions for Problem Set Seven

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1. **Page 86, number 13**

   Note that x is always positive, so $\frac{1}{x}$ and $\frac{1}{x+1}$ are always defined. We will use the first two composition solutions for the third and fourth compositions.

$$g \circ f(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{\frac{x}{x+1}} = \frac{x+1}{x} \tag{1}$$

$$f \circ g(x) = f(g(x)) = \frac{g(x)}{g(x)+1} = \frac{\frac{1}{x}}{\frac{1}{x}+1} = \frac{\frac{1}{x}}{\frac{1+x}{x}} = \frac{1}{1+x} \tag{2}$$

$$h \circ g \circ f(x) = h(g \circ f(x)) = h\left(\frac{x+1}{x}\right) = \frac{x+1}{x} + 1 = 2 + \frac{1}{x} \tag{3}$$

$$f \circ g \circ h(x) = (f \circ g)(h(x)) = \frac{1}{1+h(x)} = \frac{1}{1+1+x} = \frac{1}{2+x} \tag{4}$$

2. **Page 86, number 19**

   (b) f and g have inverses because they are bijections from $S_5$ to $S_5$. H is neither one-to-one nor onto. There is no element in h which maps to 5 (so it's not onto) and $h(1) = h(2)$, so it's not one-to-one.

   The inverses of f and g can be found by inverting each ordered pair. We can then put the pairs in order from least to greatest for reading simplicity.

$$f^{-1} = \{(1,2),(2,1),(3,5),(4,3),(5,4)\}$$

$$g^{-1} = \{(1,3),(2,4),(3,1),(4,5),(5,2)\}$$

(c) We can compose two functions piecewise. For instance $f^{-1} \circ g^{-1}(1) = f^{-1}(g^{-1}(1)) = f^{-1}(3) = 5$. So $f^{-1} \circ g^{-1}$ contains the pair $(1, 5)$.
We find the other elements and $g^{-1} \circ f^{-1}$ in the same fashion:

$$f^{-1} \circ g^{-1} = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}.$$
$$g^{-1} \circ f^{-1} = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}.$$

Using $(f \circ g)^{-1}$ from the back of the book, we can see that with these functions $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. Later, you will prove this identity for all bijections f and g which can be composed together.

3. **Page 87, number 23**

(b) Suppose $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Let $f : A \to B$ be a bijection such that $f = \{(1, 2), (2, 1)\}$. Define $g : B \to A$ such that $g = \{(1, 2), (2, 1), (3, 1)\}$. Note that g is not one-to-one because $g(2) = g(3)$. But $g \circ f : A \to A$ is the identify function on A. That is, $g \circ f = \{(1, 1), (2, 2)\}$, which is one-to-one. Thus the converse of the original statement is false.

(c) We will prove this by contradiction. Suppose $g \circ f$ is one-to-one and $f$ is not one-to-one. Then $\exists\, a_1, a_2$ where $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. "g" is a function, so $g(f(a_1)) = g(f(a_2)) \iff g \circ f(a_1) = g \circ f(a_2)$. But that means that $g \circ f$ is not one-to-one, which contradicts our assumption. Thus, f must be one-to-one.

4. **Page 87, number 24**

(a) Take $f : B \to C$ and $g : A \to B$ such that f and g are onto. We will prove that the composition, $f \circ g : A \to C$ must also be onto.

Take an element $c_0 \in C$. Since f is onto, there must exist an element $b_0 \in B$ such that $f(b_0) = c_0$. However, $b_0 \in B$. Since g is also onto, there must an exist element $a_0 \in A$ such that $g(a_0) = b_0$.

But $f \circ g(a_0) = f(g(a_0)) = f(b_0) = c_0$. Thus, for any $c \in C$ we can find an element $a \in A$ such that $f \circ g(a) = c$. So $f \circ g$ is onto.

(c) Suppose $g \circ f : A \to C$ is onto for some $f : A \to B$ and $g : B \to C$. Since $g \circ f : A \to C$ is onto for all $c_0 \in C$, $\exists\, a_0 \in A$ such that $g \circ f(a_0) = c_0$. But f is a function, so $f(a_0)$ must equal some

$b_0 \in B$. But that means that $c_0 = g \circ f(a_0) = g(b_0)$. Thus we have found a $b_0 \in B$ s.t. $g(b_0) = c_0$. Hence, g is onto.

5. **Page 87 number 27**

Note that the denominator of f is always positive. Hence

$$f(x) = \begin{cases} positive & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ negative & \text{if } x < 0, \end{cases}$$

(a) Show that f is one-to-one. Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{R}$. If $f(x) = 0$, then $x = y = 0$.

So we'll assume $f(x) \neq 0, \implies x \neq 0$ and $y \neq 0$. $f(x) = f(y) \implies \frac{x}{\sqrt{x^2+2}} = \frac{y}{\sqrt{y^2+2}}$. Since $x \neq 0, y \neq 0$, we have $\sqrt{x^2+2} = \sqrt{y^2+2}$, which implies $x^2 + 2 = y^2 + 2$. Thus $f(x) = f(y) \implies x^2 = y^2$. But we know f(x) preserves the sign of x. So this implies x = y.

Thus f is one-to-one.

(b) $\forall r \in \mathbb{R}$, $r^2 \leq r^2 + 2$. So $\sqrt{r^2} \leq \sqrt{r^2+2}$. But $\sqrt{r^2} = |r|$. So $\forall r \in \mathbb{R}$, $|r| \leq \sqrt{r^2+2}$. Thus $\frac{|r|}{\sqrt{r^2+2}} \leq 1$. We know f preserves sign, so the largest possible range of f is -1 to 1. One can verify through the following inverse that the exists a value $x \in \mathbb{R}$ such that $f(x) = r$ for any $-1 < r < 1$.

A graph of f clearly shows its range is at least -1 to 1.

(c) To find an inverse, we will set the function equal to a constant k, and solve for x. Suppose $k = f(x)$. Then $k = \frac{x}{\sqrt{x^2+2}} \Rightarrow k^2(x^2+2) = x^2$. So then $2k^2 = x^2 - k^2x^2 \Rightarrow x^2 = \frac{2k^2}{(1-k^2)}$. Thus $x = \pm\sqrt{\frac{2k^2}{1-k^2}}$. We know that f(x) preserves sign, so its inverse must also preserve sign. Note that if k is 0, the inverse is 0. Also note that since the range of f is -1 to 1, the inverse is only valid for $-1 < k < 1$.

$$f^{-1}(k) = \begin{cases} \sqrt{\frac{2k^2}{1-k^2}} & \text{if } 0 \leq k < 1, \\ -\sqrt{\frac{2k^2}{1-k^2}} & \text{if } -1 < k < 0. \end{cases}$$