Math 228: Solutions for Problem Set Five

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1. **Page 68, exercise 1 parts c,d and e**

   (c) Not a partial order. Partial orders must be antisymmetric. If $a = 2$ and $b = -2$, then $a^2 = b^2 = 2$. So then $a^2 \leq b^2$ and $b^2 \leq a^2$, which implies $a \preceq b$ and $b \preceq a$. But $a \neq b$. Since this is not a partial order, it is not a total order.

   (d) Not a partial order. Partial orders must be antisymmetric. If $(a, b) = (1, 2)$ and $(c, d) = (1, 3)$, then $(a, b) \preceq (c, d)$ since $a \leq c$ and $(c, d) \preceq (a, b)$ since $c \leq a$. But $(a, b) \neq (c, d)$. Since this is not a partial order, it is not a total order.

   (e) R is a partial order. To prove this we must prove that this relation is reflexive, antisymmetric and transitive.

      i. Reflexive. Take $(a, b) \in \mathbb{N} \times \mathbb{N}$. $a \leq a$ and $b \geq b$. So then $(a, b) \preceq (a, b)$. Thus R is reflexive.

      ii. Antisymmetric. Suppose $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$ for some $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$. Then $a \leq c$ and $c \leq a$, so $a = c$. Similarly, $b \geq d$ and $d \geq b$, so $b = d$. Since $a = c$ and $b = d$, $(a, b) = (c, d)$. Hence, R is antisymmetric.

      iii. Transitive. Suppose $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$ for some $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$. Then $a \leq c$ and $c \leq e$, so $a \leq e$. Similarly, $b \geq d$ and $d \geq f$, so $b \geq f$. Since $a \leq e$ and $b \geq f$, $(a, b) \preceq (e, f)$. So R is transitive.

   Thus the relation is a partial order.

   However, this relation is not a total order. Suppose $(a, b) = (1, 1)$ and $(c, d) = (2, 2)$. Since $b \not\geq d$, $(a, b) \not\preceq (c, d)$. Since $c \not\leq a$, $(c, d) \not\preceq (a, b)$. So then (a, b) and (c, d) are not comparable and R is not a total order.

2. **Page 69, number 5**

   (a)
   $$\begin{array}{c} 6 \\ | \\ 5 \\ | \\ 4 \\ | \\ 3 \\ | \\ 2 \\ | \\ 1 \end{array}$$

3. **Page 69, number 6**

   (a) From 5a, "6" is the maximum and the only maximal value (maximum values are always maximal). "1" is the minimum and the only minimal value.

   (b) From 5b, $\{c, d\}$ and $\{a\}$ are not supersets to any set in the group, so they are both minimal. However, neither is a subset to the other, so there is no minimum in this relation. $\{a, b, c, d\}$ is a superset to all other sets in the group, so it is both the maximum and maximal.

4. **Page 69, number 10**

   (a) To prove that R is a partial order, we must prove it is antisymmetric, reflexive and transitive.

      i. Reflexive. Take $a = (a_1, a_2) \in \mathbb{Z}^2$. We know $a_1 \leq a_1$ and $a_1 + a_2 \leq a_1 + a_2$. So $a \preceq a$. Hence R is reflexive.

      ii. Antisymmetric. Suppose $a \preceq b$ and $b \preceq a$ for some $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{Z}^2$. Then $a_1 \leq b_1$ and $b_1 \leq a_1$, so $a_1 = b_1$. But we also know that $a_1 + a_2 \leq b_1 + b_2$ and $b_1 + b_2 \leq a_1 + a_2$, which implies $a_1 + a_2 = b_1 + b_2$. But since $a_1 = b_1$, $a_2 = b_2$. Hence $a = b$ and R is antisymmetric.

      iii. Transitive. Suppose $a \preceq b$ and $b \preceq c$ for some $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$ in $\mathbb{Z}^2$. Then $a_1 \leq b_1$ and $b_1 \leq c_1$, so $a_1 \leq c_1$. But we also know that $a_1 + a_2 \leq b_1 + b_2$ and $b_1 + b_2 \leq c_1 + c_2$, which implies $a_1 + a_2 \leq c_1 + c_2$. Hence $a \preceq c$ and R is transitive.

This relation is not a total order. Suppose $a = (1, 10)$ and $b = (2, 3)$. Then $b_1 \not\leq a_1$ so $b \not\preceq a$. Also, $a_1 + a_2 = 11$ and $b_1 + b_2 = 5$, so $a_1 + a_2 \not\leq b_1 + b_2$ so $a \not\preceq b$. But a total order requires that all pairs of elements are comparable. So this is not a total order.

(b) We'll construct a generalization using sigma notation. Suppose $a, b \in \mathbb{Z}^n$ such that $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$. $a \preceq b$ if and only if $\forall i$ where $1 < i < n$,

$$(\sum_{j=1}^{i} a_j) \leq \sum_{j=1}^{i} b_j$$

.

5. **Problem 5 on handout**

(b) See part c for explanation. $glb(12, 16) = 4$ and $lub(12, 16) = 48$.

(c) (a, b) is in the relation if and only if $\frac{a}{b}$ is an integer, which means (a, b) is in the relation if and only if a divides b.

The glb of two numbers a and b in this relation is a number, $c \in \mathbb{N}$ for which $c \mid a$ and $c \mid b$. Logically if c is the greatest common divisor of a and b, the following statement also holds: if $\exists d \in \mathbb{N}$ s.t. $d \mid a$ and $d \mid b$ then d must divide c. But this statement only holds true for one number, so $c = gcd(a, b)$.

Similarly, the lub of a and b is a number $c \in \mathbb{N}$ such that $a \mid c$ and $b \mid c$, so c is a multiple of both a and b. Additionally, if $\exists d \in \mathbb{N}$ s.t. $a \mid d$ and $b \mid d$ then d must divide c. But a and b both divide lcm(a,b), so lcm(a,b) must divide c. Since c is also a multiple of a and b, by definition of lcm, it must equal lcm(a,b).