Math 228: Solutions for Problem Set Three

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Note: Grading weights have been changed for this assignment. Problems requiring proofs have been given higher weight. The total points on this homework remains 25.

1. **Page 50, number 28 – True or False**

   (a) True. Suppose $(a, b) \in A \times B$ for some $a \in A$ and $b \in B$. Since $A \subseteq C$, $a \in A \Rightarrow a \in c$. Similarly, $b \in B \Rightarrow b \in D$. So $(a, b) \in C \times D$. Thus any element in $A \times B$ is also in $C \times D$. So $A \times B \subseteq C \times D$.

   (b) False. Suppose $A = \{1, 2\}$, $B = \{1\}$ and $C = \{1, 2, 3\}$.
   Then $A \nsubseteq B$, $B \subseteq C$ but $A \subseteq C$.

   (c) False. Suppose $A = \{a, b\}$, $B = \emptyset$, $C = \{a\}$ and $D = \{a\}$. Then $A \times B = \emptyset \subseteq C \times D$. But $A = \{a, b\} \nsubseteq \{a\} = D$. So the supposition is false. However, it does hold if we know that neither A nor B is the empty set:

   Suppose $(a, b) \in A \times B$ such that $a \in A$ and $b \in B$. If $A \times B \subseteq C \times D$ then $(a, b) \in C \times D$. So then, $a \in C$ and $b \in D$. But a could have been anything in A and b could have been anything in B. Hence, $A \subseteq C$ and $B \subseteq B$.

   (d) False. This follows directly from part (c).

   (e) True. Suppose $x \in A$. Then, $x \in A \cup B$. Since $A \cup B \subseteq A \cap B$, $x \in A \cap B$. But then $x \in B$. Hence, $A \subseteq B$. Similarly, $B \subseteq A$. So $A = B$.

2. **Page 51, number 30 – More true/false**

   (a) False. Let $A = \{1, 2, 3\}$, $B = \{1\}$ and $C = \{2\}$.
   $A \setminus (B \cup C) = \{3\}$. However, $(A \setminus B) \cup (A \setminus C) = \{2, 3\} \cup \{1, 3\} = \{1, 2, 3\}$.

   (b) True. Suppose $(a, c) \in (A \setminus B) \times C$. Then $a \in A \setminus B$. Since $a \in A$, $(a, c) \in A \times C$, but since $a \notin B$, $(a, c) \notin B \times C$. So, $(a, c) \in (A \times C) \setminus (B \times C)$. Thus $(A \setminus B) \times C \subseteq (A \times C) \setminus (B \times C)$.

   On the other hand, take $(a, c) \in (A \times C) \setminus (B \times C)$. Then $(a, c) \in A \times C$, but $(a, c) \notin B \times C$. Now $(a, c) \in A \times C$ implies $a \in A$ and $c \in C$. But $c \in C$ and $(a, c) \notin B \times C$ implies $a \notin B$. So $a \in A \setminus B$. Therefore $(a, c) \in (A \setminus B) \times C$. Thus, $(A \times C) \setminus (B \times C) \subseteq (A \setminus B) \times C$. Hence, $(A \times C) \setminus (B \times C) = (A \setminus B) \times C$

   (c) True. See solution on page S-9 in the back of the book.

   (d) False. Let $A = \{1\}$, $B = \{2\}$, $C = \{1\}$ and $D = \{1\}$.
   $(A \cup B) \times (C \cup D) = \{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
   $(A \times C) \cup (B \times D) = \{1, 1\} \cup \{2, 2\} = \{(1, 1), (2, 2)\}$

   (e) False. Let $A = \{1\}$, $B = \{2\}$, $C = \{2\}$ and $D = \{2\}$.
   $A \setminus C = \{1\}$ and $B \setminus D = \emptyset$, so $(A \setminus C) \times (B \setminus D) = \emptyset$
   However, $A \times C = \{(1, 2)\}$ and $B \times D = \{(2, 2)\}$, so $(A \times C) \setminus (B \times D) = \{(1, 2)\}$.

3. **Page 56, number 6**

A binary relationship R is symmetric if $(a, b) \in R \Longrightarrow (b, a) \in R$. A binary relationship R is antisymmetric if $(a, b) \in R, (b, a) \in R \Longrightarrow a = b$.

   Take $(a, b) \in R$. From the symmetric law, if $(a, b) \in R$ then $(b, a) \in R$. But then, from the antisymmetric law, $a = b$. So, $\forall (a, b) \in R, a = b$. Hence, either R is composed of pairs (a, a) where $a \in A$ or R is the empty set.

4. **Page 57, number 9, part j**

   (a) Reflexive: $\forall a \in \mathbb{N}, \frac{a}{a} = 1$, which is an integer, so $(a, a) \in R$.

   (b) Not symmetric: $\frac{2}{1} = 2$, so $(2, 1) \in R$, but $\frac{1}{2} \notin \mathbb{Z}$ so $(1, 2) \notin R$.

   (c) Antisymmetric: Suppose $(a, b), (b, a) \in R$ for some $a, b \in \mathbb{N}$. Then $\frac{a}{b}$ and $\frac{b}{a}$ are positive integers. Since $\frac{a}{b}$ is an integer, $b \mid a$. Similarly, since $\frac{b}{a}$ is an integer, $a \mid b$. We know a and b are both positive, so they must be equal.

(d) Transitive: Suppose $(a,b),(b,c) \in R$. Then $\frac{a}{b}$ and $\frac{b}{c}$ are positive integers. But the product of two integers is an integer. So $\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c}$ is an integer. Hence $(a,c) \in R$.

5. **Page 57, number 11**

(a) i. Reflexive: For any set X, $X \subseteq X$, so $(X,X) \in R$.

ii. Not symmetric: Take $X = \emptyset$ and $Y = \{a\}$. Then $X \subseteq Y$, so $(X,Y) \in R$ but $Y \nsubseteq X$, so $(Y,X) \notin R$.

iii. Antisymmetric: Suppose $(X,Y),(Y,X) \in R$. Then $X \subseteq Y$ and $Y \subseteq X$. Thus $X = Y$.

iv. Transitive: Take $(X,Y),(Y,Z) \in R$. Then $X \subseteq Y$ and $Y \subseteq Z$. Therefore $X \subseteq Z$. Hence $(X,Z) \in R$.

(b) i. Not reflexive: For any set X, X is not a proper subset of itself, so $(X,X) \notin R$.

ii. Not symmetric: Take $X = \emptyset$ and $Y = \{a\}$. Then $X \subsetneq Y$, so $(X,Y) \in R$ but Y is not a proper subset of X, so $(Y,X) \notin R$.

iii. Antisymmetric: Suppose $(X,Y),(Y,X) \in R$. Then $X \subsetneq Y$ and $Y \subsetneq X$. This is a contradiction, so our supposition was wrong. But that means that there are never both $(X,Y) \in R$ and $(Y,X) \in R$ for a given X and Y. So antisymmetric vacuously holds.

iv. Transitive: Take $(X,Y),(Y,Z) \in R$. Then $X \subsetneq Y$ and $Y \subsetneq Z$. Therefore $X \subsetneq Z$. Hence $(X,Z) \in R$.

(c) i. Not reflexive: Let $X = \{a\}$. $X \cap X = \{a\} \neq 0$. So $(X,X) \notin R$.

ii. Symmetric: Suppose $(X,Y) \in R$. Then $X \cap Y = \emptyset$. Thus $Y \cap X = \emptyset$ and $(Y,X) \in R$.

iii. Not antisymmetric: Let $X = \{a\}$ and $Y = \emptyset$. Then $X \cap Y = \emptyset$, so $(X,Y),(Y,X) \in R$. But $X \neq Y$.

iv. Not transitive: Let $X = \{a\}$, $Y = \{b\}$ and $Z = \{a\}$. Then $X \cap Y = \emptyset$ and $Y \cap Z = \emptyset$ so $(X,Y),(Y,Z) \in R$. But $X \cap Z = \{a\}$, so $(X,Z) \notin R$.