# Math 228: Solutions for Problem Set Eighteen

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1. **Page 215, number 7**

   (c) Represent the possible lotto numbers with set $N = \{1, 2, \ldots, 49\}$. We can partition $N$ into two disjoint sets: $L \subseteq N$ are the numbers picked in the lottery drawing and $S \subseteq N$ are the numbers not picked in the lottery drawing. By definition, $L \cup S = N$, $L \cap S = \emptyset$, $|L| = 6$ and $|S| = 43$. For each of the following scenarios, we will pick some numbers from L and some numbers from S.

      i. All six of our numbers come from S. So the possible combinations are $\binom{|S|}{6} = \binom{43}{6}$.

      ii. One number comes from L and the remaining numbers come from S. So $\binom{|L|}{1}$ represents the choice for the matching number and the remaining 5 numbers are chosen from S. The the total possible combinations is $\binom{6}{1}\binom{43}{5}$.

      iii. Two numbers come from L, and 4 numbers come from S. $\binom{6}{2}\binom{43}{4}$.

      iv. Three numbers come from L, and 3 numbers come from S. $\binom{6}{3}\binom{43}{3}$.

      v. Four numbers come from L, and 2 numbers come from S. $\binom{6}{4}\binom{43}{2}$.

      vi. Five numbers come from L, and 1 number comes from S. $\binom{6}{5}\binom{43}{1}$.

      vii. All six numbers come from L. Intuitively, there should be only one combination. $\binom{6}{6} = 1$.

   (d) The previous part contains every possibility for a ticket in the lottery. So the sum should be equal to the choice of all possible tickets, or $\binom{49}{6}$. In fact, this summation fits the identity in problem 3!

2. **Page 216, number 15**

   (a) This is a standard combination. $\binom{9}{6} = 84$

(b) There are two intuitive ways to do this:

   i. In how many ways were both angry people invited from part a? If they were both invited, there are four remaining spots with seven possible people, so $\binom{7}{4}$. The number of ways in which they'd not both be invited is $\binom{9}{6} - \binom{7}{4} = 49$.

   ii. Remove person A and person B from the group (the ones who can't get along). We can break the problem up into three scenarios for invitations: invite person A, invite person B, or invite neither A nor B.

   In the first scenario, we are inviting person A. We have 5 spots left to fill and cannot invite person B. So there are $\binom{7}{5}$ possibilities. Similarly, there are $\binom{7}{5}$ possibilities when we invite person B.

   Now suppose we invite neither A nor B. There are 7 people left to choose from, and 6 spots to fill. So this scenario has $\binom{7}{6}$ possibilities.

   Together there are $\binom{7}{6} + 2 \cdot \binom{7}{5} = 49$ choices.

(c) Separate the group into married couples and unmarried couples. We can either invite 3 married couples or 2 couples. We cannot invite one or zero couples because then we will be unable to fill our dinner party.

   If we're inviting three married couples, we have no choices – we must invite all three couples and no singles. So there is 1 combination for this scenario.
   If we're inviting two married couples, we have $\binom{3}{2}$ choices for which couples to pick and $\binom{3}{2}$ choices for which two singles to pick.

   So the total number of combinations is $1 + \binom{3}{2}\binom{3}{2} = 10$.

3. **Question 3 on Problem Set**

   Since our set is composed of m oranges and n apples, any r combination must contain some number of oranges and the remaining items must be apples. Suppose there are i oranges in the combination. Then there are $r - i$ apples.

If we fix i and r, the total number of combinations is $\binom{m}{i}\binom{n}{r-i}$, where we choose i oranges and $r-i$ apples.

But every possible combination must have some number of oranges between 0 and r. So the total number of r combinations from the set is:

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

4. **Question 4 on Problem Set**

   Using the identity from Question 3 with $m = r = n$, we get

$$\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$$

   However, $\binom{n}{n-i} = \binom{n}{i}$. So,

$$\begin{aligned}\binom{2n}{n} &= \sum_{i=0}^{n}\binom{n}{i}\binom{n}{i}\\ &= \sum_{i=0}^{n}\binom{n}{i}^2\\ &= \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n-1}^2 + \binom{n}{n}^2\end{aligned}$$

5. **Question 5 on Problem Set**

   We will prove this is the Fibonacci sequence by proving it has the same recurrence relation. Certainly $G_1 = F_1$ and $G_2 = F_2$. We now must prove that for all $n > 2$, $G_n = G_{n-1} + G_{n-2}$.

   We know that:

$$G_{n+1} = \binom{n}{0} + \binom{n-1}{2} + \binom{n-2}{3} + \binom{n-3}{4} + \ldots$$

   Let's look at $G_{n-1}$ and $G_n$.

$$G_{n-1} = \binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2} + \ldots \quad (1)$$

$$G_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \binom{n-4}{3} + \ldots \quad (2)$$

In the first column, $\binom{n-2}{0} = 1 = \binom{n}{0}$.

Using the identity $\binom{j}{k} = \binom{j-1}{k-1} + \binom{j-1}{k}$, we can add column by column, and match the sum of each column to the above term in $G_{n+1}$.

Thus $G_{n+1} = G_n + G_{n-1}$, and so $G_n$ is the $n^{th}$ Fibonacci number.