Math 228: Solutions for Problem Set Sixteen

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1. **Page 191, number 5**

   Consider the set S of students of who entered the store. We know $|S| = 15$. Let $P \subseteq S$ consist of the students of bought popsicles, $G \subseteq S$ be the students who bought gum, and $C \subseteq S$ be the students who bought candy bars. We know $P \cup G \cup C \subseteq S$, so then $|P \cup G \cup C| \leq 15$.

   However, from inclusion-exclusion principle,

   $$|P \cup G \cup C| = |P| + |G| + |C| - |G \cap P| - |P \cap C| - |C \cap G| + |P \cap G \cap C|$$

   From other information in the problem this means that,

   $$|P \cup G \cup C| = 16 + |P \cap G \cap C|$$

   So then $16 + |P \cap G \cap C| \leq 15$. This is obviously not true, so there must be a bookkeeping problem, or the clerk made an error and mischarged somebody.

2. **Page 192, number 16**

   Let N be the set of integers between 1 and 10^6. Let $S \subseteq N$ be the set of perfect squares inside N and $C \subseteq N$ be the set of perfect cubes inside N.

   We know that $(10^2)^3 = 10^6$, so 10^2 is the last integer whose cube is inside 10^6. Alternatively, $\sqrt[3]{10^6} = 10^2$. Thus there are $10^2 = 100$ perfect cubes inside N and $|C| = 100$.

   Similarly, we know that $(10^3)^2 = 10^6$, so 10^3 is the last integer whose square is inside N so $|S| = 1000$.

Finally, from the hint, we know that any number which is both a perfect square and a perfect cube is also a perfect sixth power (i.e. their sixth root is an integer). This is precisely the set $S \cap C$. We know 10 is the last number whose sixth power is inside 10^6. So $|S \cap C| = 10$.

By the inclusion-exclusion principle, $|S \cup C| = |S| + |C| - |S \cap C|$, so $|S \cup C| = 1090$. This is the number of integers which **are** perfect squares or cubes. We want the complement of this set.

$$|(S \cup C)^c| = |N| - |S \cup C| = 10^6 - 1090$$

3. **Page 198, number 16**

   Note: I interpreted this problem differently from the book's answer in part (a). I ignored the order in which your cards are dealt to you. I will provide solutions for both interpretations and will not take off points for either interpretation, provided you did not switch between the two without explanation.

   (b) i. Counting order. The pairs can come in three different orders: AABB, ABBA or ABAB. For each order, there are 52 choices for the first card, 3 for the second of that pair, 48 first care in the second pair, and 3 for the second card of the second pair. So there are $3 \cdot (52 \cdot 3 \cdot 48 \cdot 3) = 67392$ possibilities.

   ii. Ignoring order. Ignoring suits, we have 13 choices for the value of the first pair and 12 for the value of the second pair. We must divide by 2 to account for swaps. Looking at suits, for each pair, there are 4 possibilities for the first card and 3 for the second card, divided by 2 to account for swaps. So there are $\frac{13 \cdot 12}{2} \cdot \frac{4 \cdot 3}{2} \cdot \frac{4 \cdot 3}{2} = 2808$ possibilities.

   (c) i. Counting order. From the (a) we know there are 312 ways for four of a kind. This set is disjoint from 2 pair (their intersection is empty) so we can just add the possibilities of both: $67392 + 312 = 67704$.

   ii. Ignoring order. Ignoring card order there are logically only 13 ways to get four of a kind. Again we can just add both sets: $2808 + 13 = 2821$.

   (d) i. Counting order. The cards can be dealt in four different ways, each with the same number of possibilities: AAAB, AABA, ABAA, BAAA. For each organization of cards, there are 52 possibilities for the first card in the triple, 3 for the second in the triple and 2 for the third in the triple. There are 48 cards for the kicker. Since there are four organizations the total possibilities are $4 \cdot 52 \cdot 3 \cdot 2 \cdot 48 = 59904$.

ii. Ignoring order. There are 13 possibilities for the suit of the triple and 4 choices for the suit of the card which is not dealt to you. There are 48 remaining cards for the kicker.

$$13 \cdot 4 \cdot 48 = 2496$$

(e) The set of hands which have at least one pair can be partitioned into four-of-a-kind hands, three-of-a-kind hands, 2-pair hands, and 1-pair hands. Each type of hand does not overlap with any other set. So we can find the size of their union by adding up all their respective sizes. We've already found the size of everything but one-pair hands.

i. Counting order. A one pair hand can be dealt in six ways where A's are pairing cards and B's are non-pairing cards: AABB, ABAB, ABBA, BABA, BAAB and BBAA. For each type of order, there are 52 possibilities for the first card in the pair, 3 possibilities for the second card in the pair. For the non pairing cards, there are 48 possibilities for the first non-pairing card. The second non-pairing card can't pair with the first non-pairing card, or with the pair, so there are 44 possibilities. So the total possibilities for one-pair hands is

$$6 \cdot 52 \cdot 3 \cdot 48 \cdot 44 = 1976832$$

To find the total possibilities for hands with at least one pair, we add this to the possibilities of the other hands:

$$1976832 + 59904 + 67392 + 312 = 2104440$$

Note that there are about 6.5 million different hands counting order, so this is about 1 in 3 probability.

ii. Ignoring order. There are 13 values for the pair and $\frac{4 \cdot 3}{2}$ choices for the suits of the cards. There are 48 possibilities for the next card and 44 possibilities for the last card since it can't pair with the previous card or the pair. We must also divide by 2 to account for swapping of the two non-pairing cards. So the total combinations for one-pair hands are

$$13 \cdot \frac{4 \cdot 3}{2} \cdot \frac{48 \cdot 44}{2} = 82368$$

To find the total possibilities for hands with at least one pair, we add this to the possibilities of the other hands:

$$82368 + 2496 + 2808 + 13 = 87685$$

There are $\frac{52 \cdot 51 \cdot 50 \cdot 49}{4!} \approx 270000$ hands ignoring order, which is about a 1 in 3 probability of getting at least a pair, as before.

4. **Page 199, number 24**

   For any s, t is completely determined. Because $gcd(s,t) = 1$, s and t cannot share any prime factors. So s is a product of all the factors of some number of primes from n, and $t = n/s$. For each prime, s can contain its all factors in n or not. So there are 2^r choices for s. However, we must divide by two, because these are unordered pairs and we must avoid duplicates. So there are 2^{r-1} unordered pairs of $\{s,t\}$.

5. **Page 134, number 20**

   Let $F := Fun(A,B)$, the set of functions from A to B.

   (a) Let $f \in F$. For each $a \in A$, $f(a)$ must go to some element in B. Thus for each element a in A, there are $|B|$ choices where f(a) can go. But a function in F is precisely determined by where the elements of A go. So

$$|F| = \overbrace{|B||B| \cdots |B|}^{|A| \text{ times}} = |B|^{|A|} = 3^n$$

   (b) A function in F which is not onto must not go to either 1, 2 or 3 – otherwise it would be onto. Let $F_{\#1} \subseteq F$ be the set of functions which does not go to 1. Similarly define $F_{\#2}$ and $F_{\#3}$. If a function is in one of these three sets, then it is certainly not onto. So a function f is not onto if and only if $f \in F_{\#1} \cup F_{\#2} \cup F_{\#3}$.

   If a function f does not go to 1, then for each $a \in A$, f(a) is 2 or 3. Thus there are 2^n possibilities for f. So $|F_{\#1}| = 2^n$. Similarly, $|F_{\#2}| = |F_{\#3}| = 2^n$.

   Now lets look at intersections of these sets. Suppose $f \in F_{\#1} \cap F_{\#2}$. Then f does not go to 1 or 2. So then $\forall a \in A$, $f(a) = 3$. Thus there

is only one element in $F_{\#1} \cap F_{\#2}$.
Similarly, $|F_{\#1} \cap F_{\#3}| = |F_{\#2} \cap F_{\#3}| = 1$.

Finally, let's look at the intersection of all three sets. Suppose $f \in F_{\#1} \cap F_{\#2} \cap F_{\#3}$. The f does not go to 1, 2 or 3. But this is impossible: f(a) must go somewhere. So $F_{\#1} \cap F_{\#2} \cap F_{\#3} = \emptyset$.

By the inclusion-exclusion principle,

$$\begin{aligned}
|F_{\#1} \cup F_{\#2} \cup F_{\#3}| &= |F_{\#1}| + |F_{\#2}| + |F_{\#3}| - |F_{\#1} \cap F_{\#2}| - |F_{\#1} \cap F_{\#3}| \\
&\quad - |F_{\#2} \cap F_{\#3}| + |F_{\#1} \cap F_{\#2} \cap F_{\#3}| \\
&= 2^n + 2^n + 2^n - 1 - 1 - 1 + 0 \\
&= 3 \cdot 2^n - 3
\end{aligned}$$

(c) The set of functions which are onto is the complement to the set of functions which **are not** onto.

$$\begin{aligned}
|(|F_{\#1} \cup F_{\#2} \cup F_{\#3})^c| &= |F| - |F_{\#1} \cup F_{\#2} \cup F_{\#3}| \\
&= 3^n - 3 \cdot 2^n + 3 \\
&= 3 \cdot (3^{n-1} - 2^n + 1)
\end{aligned}$$